Sparse Navigable Graphs for Nearest Neighbor Search

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University of Pennsylvania

(SODA 2026)

Outline of the Talk

1. **Background** on navigability

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Concurrent work with overlapping results!

- Conway, Dhulipala, Farach-Colton, Johnson, Landrum, Musco, Schechter, Suel, Wen. *Efficiently Constructing Sparse Navigable Graphs*. (2025)

Background

(1/4)

Nearest Neighbor Search

Nearest Neighbor Search (NNS)

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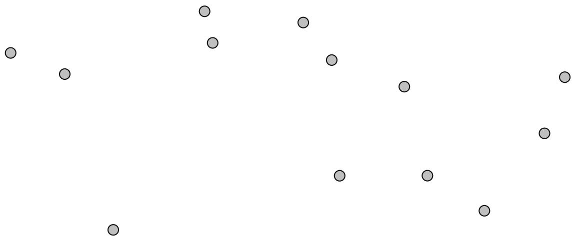

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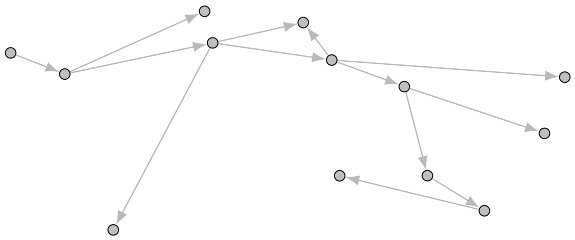

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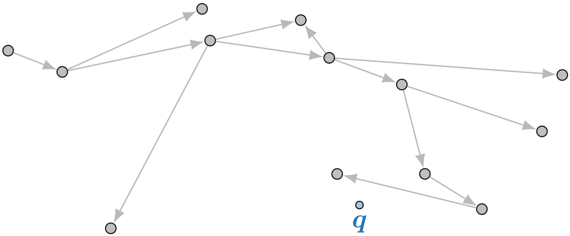

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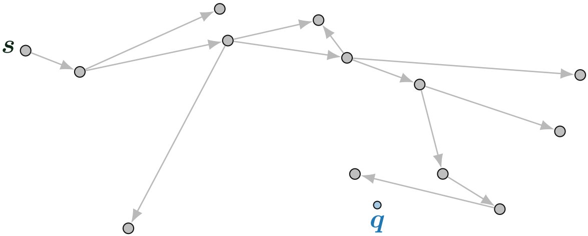

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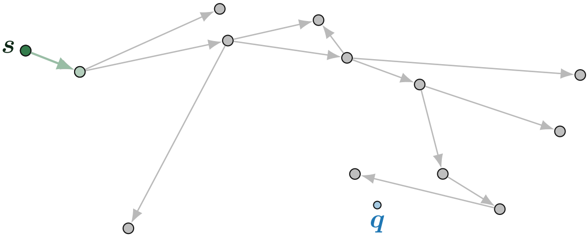

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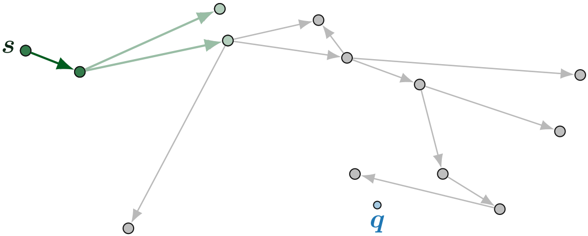

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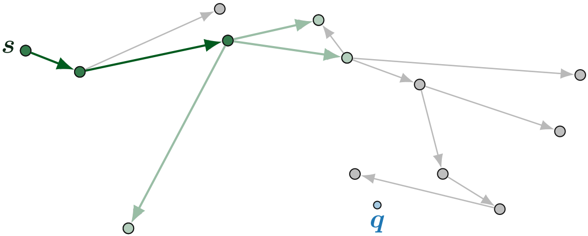

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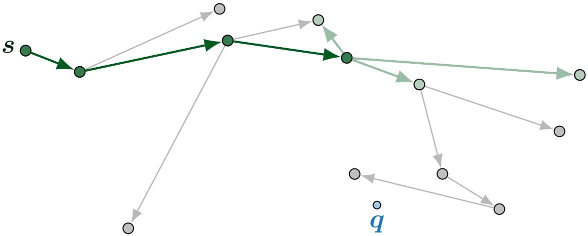

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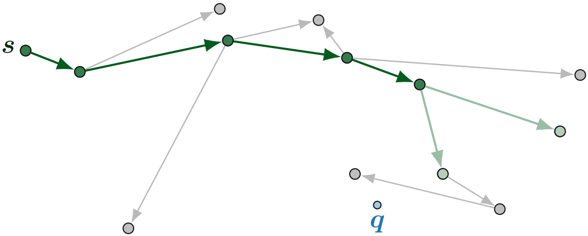

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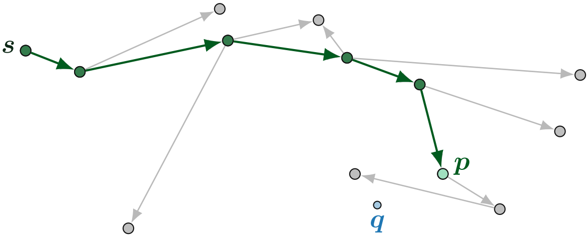

Why Graphs? It's a Small World

Real-world networks exhibit *small world* behavior

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Real-world networks exhibit *small world* behavior

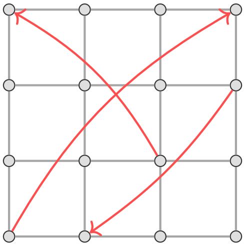
Milgram (1969). Packages from Nebraska $\longrightarrow$ Boston

- "Pass to a friend closer to Boston"
- Median chain length: only six!

Navigability in Small-World Graphs

Kleinberg (2000). Formalization of Milgram's experiment

- **Model:** Grid + random long-range edges
- **Rule:** Move to neighbor closest to the target



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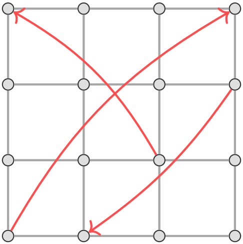

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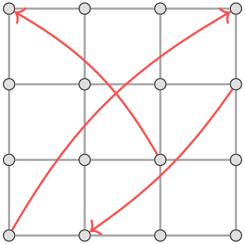

Q. Can we make *arbitrary data* navigable for NNS?

(Motivation behind modern NNS heuristics!)

Navigability in General Metrics

Dataset $\mathbf{P}$ with metric d; directed graph $\mathbf{G} = (\mathbf{P}, E)$.

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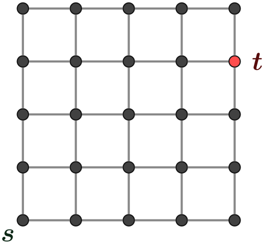

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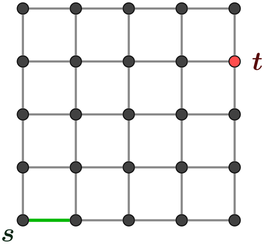

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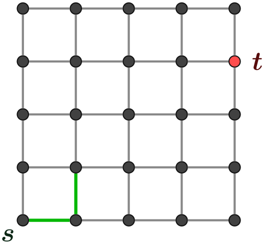

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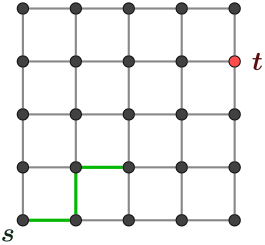

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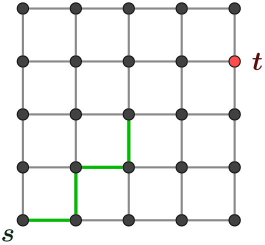

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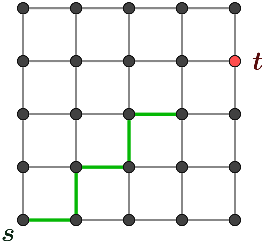

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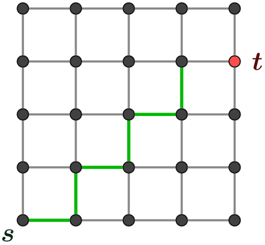

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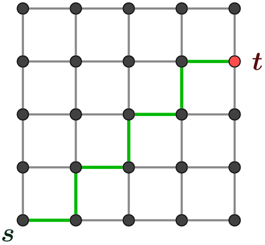

Nearest Neighbor Search?

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Approximate Nearest Neighbor (ANN)

Return *any* c-approximate nearest neighbor $\boldsymbol{p}$ of query $\boldsymbol{q}$:

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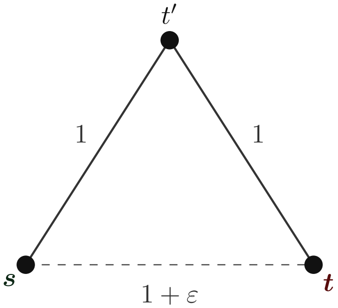

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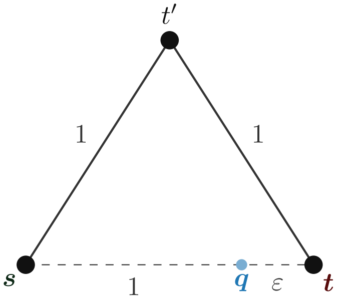

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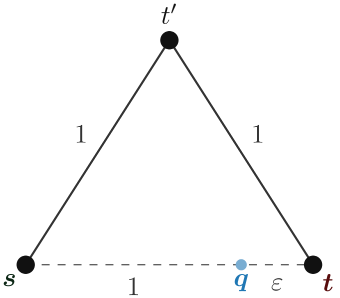
$\mathrm{GreedySearch}(\boldsymbol{s}, \boldsymbol{q}) \to (1/\varepsilon)$-ANN



α-Navigability: a small (world) fix!

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α-Navigability [Indyk-Xu, 2023]

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Surprising Theorem! [Indyk–Xu, 2023]

$\forall \alpha > 1$, if $\mathbf{G}$ is α-navigable, then $\mathrm{GreedySearch}$ returns a

$$\left(\frac{\alpha+1}{\alpha-1} + \varepsilon\right)\text{–ANN in } O(\log(\Delta/\varepsilon)) \text{ hops.}$$

Takeaway: sparse α-navigable graphs $\implies$ fast ANN!

Building α-Navigable Graphs

Heuristics: DiskANN, HNSW, etc.

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Our Paper:

Sparsest Navigable Graph (SNG)

Given dataset $(\mathbf{P}, \mathrm{d})$ and $\alpha \geq 1$,

- What is the sparsest α-navigable graph on $\mathbf{P}$?
- How fast can we compute (or approximate) it?

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Sparsity objective: minimize maximum degree

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($\lambda(P) :=$ doubling dimension of P)

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Result 1

There is a dataset $\mathbf{P}$ where:

- $\exists\ 1$-navigable graph of max-degree $O(\log n)$
- Slow-DiskANN outputs graph of degree $\Theta(n)$

$\Longrightarrow$ Slow-DiskANN gives $\tilde{\Omega}(n)$-approximation (very bad!)

Set Cover View

(2/4)

SNG $\longrightarrow$ Set Cover

α-Navigability from source s

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Equivalent Set Cover instance.

Elements	$\mathbf{P} \setminus \{s\}$
Sets	$Z(s, u) := \{t \mid \mathsf{d}(u, t) < \mathsf{d}(s, t)/\alpha\}$
Cover size	$\deg(s)$

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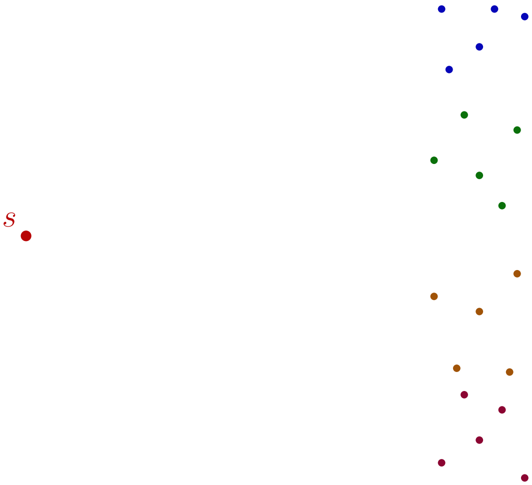
Key: α-navigability $\equiv$ n Set Cover instances

SNG $\longrightarrow$ Set Cover



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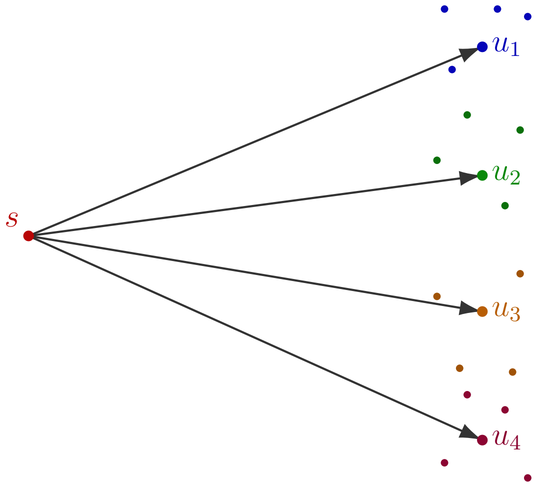

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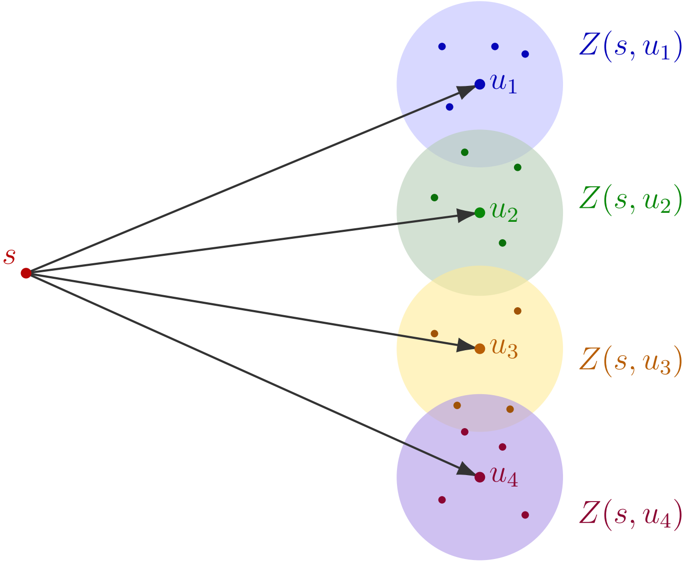

Algorithm via Greedy Set Cover

Key: α-navigability $\equiv$ n Set Cover instances

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Each instance:

- $n-1$ elements, $n-1$ sets

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- $n-1$ elements, $n-1$ sets
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Result 2A

$O(n^3)$-time algorithm for $(\ln n + 1)$-approximation to SNG.

Hardness via Set Cover

Result 2B

NP-hard to compute $(c \ln n)$-approximation to SNG, for some $c > 0$.

Idea. Encode Set Cover as a navigability condition!

Baseline Results via Set Cover

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Q. Can we compute $O(\ln n)$-approximation in time $o(n^3)$?

Faster Algorithms

(3/4)

Membership Set Cover

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Set Cover (Membership Model)

Access input only via queries “is element x in set S?”

Algorithm via Membership Set Cover

n elements, m sets, $k =$ min cover size

Lemma

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- $\mathrm{OPT}_{\mathsf{size}} = \tilde{O}(n) \implies \tilde{O}(n^2)$ runtime
- $\alpha = 1 \implies \mathrm{OPT}_{\mathsf{size}} = \tilde{O}(n^{1.5})$ [DGM+24]

  $\implies \tilde{O}(n^{2.5})$ runtime

Fast Algorithm for Membership Set Cover

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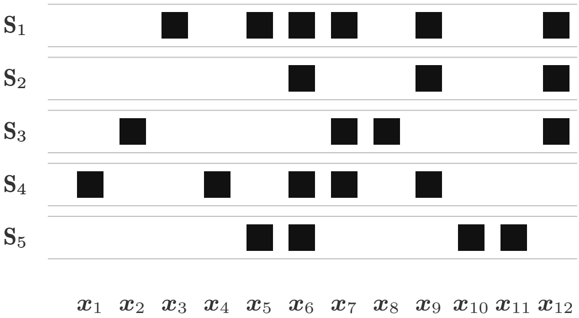
Idea. Simulate greedy via random sampling

- Greedy: heaviest set covers $\geq (1/k)$-fraction of elements

FindHeavySet: Illustration



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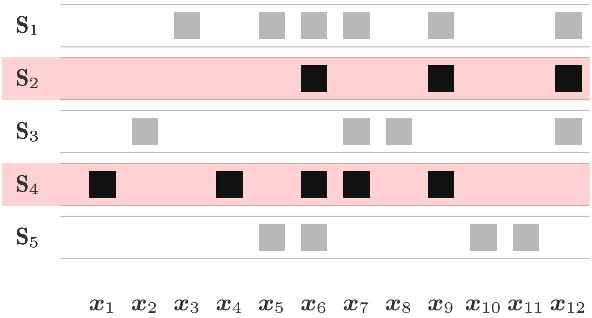

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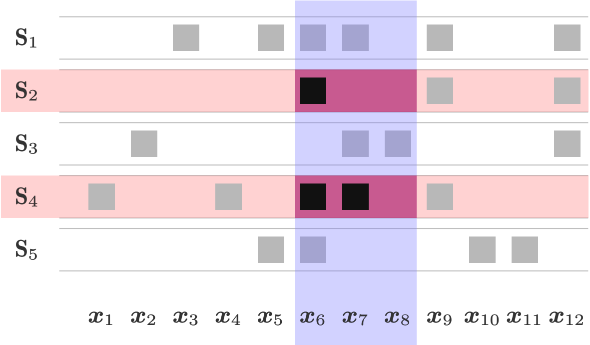

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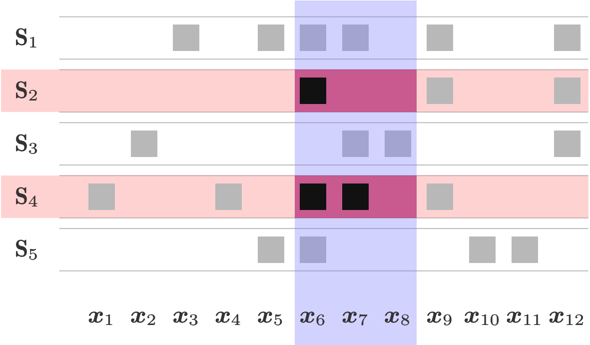

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- Set sample: $\{S_2, S_4\}$
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- S_4 hits many elements $\Longrightarrow$ good set to pick!

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Goal. Find a set covering $\Omega(1/k)$-fraction of uncovered elements

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vs.

$\Omega(mn)$ to find heaviest

Final Algorithm

Call FindHeavySet until full cover $\longrightarrow O(k \ln n)$ rounds

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Q. What if $\mathrm{OPT}_{\text{size}} = \tilde{\Omega}(n^2)$?

Bicriteria Approximation to SNG

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Given $\alpha \le \beta$, build α-navigable graph $\mathbf{G}$.

- Let $k_\beta :=$ max-degree of sparsest β-navigable graph
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Result 4 (informal)

$\tilde{O}(n^\omega)$-time algorithm for $O(\ln n)$-approximation to $(\alpha, 2\alpha)$-SNG.

($\omega \approx 2.37 =$ matrix multiplication exponent)

Verification via Matrix Multiplication

How to even verify that $\mathbf{G}$ is α-navigable?

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Naive:

- For all $s \neq t \in \mathbf{P}$ and $(s, u) \in E$, check

$$\mathsf{d}(u, t) < \mathsf{d}(s, t)/\alpha$$

- $\Theta(n^2 \cdot \deg(\mathbf{G})) \longrightarrow$ potentially $\Omega(n^3)$ time!

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Faster Algorithms for SNG

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Q. Even faster algorithms for worse approximation?

Lower Bound

(4/4)

Simple $\Omega(n^2)$ Lower Bound

Result 5

$\Omega(n^2)$ queries to $\mathsf{d}(\cdot,\cdot)$ needed for *any* $o(n)$-approximation to SNG.

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Idea. Navigability $\implies$ graph contains minimum-distance edge

- Fix a metric with constant-degree navigable graph
- Shrink a random distance $\longrightarrow$ hidden shortcut

Perturbed Path Metric

Metric on $[n]$:

$$\mathsf{d}(i,j) = 1 + \frac{|i-j|}{n-1}$$

Perturbation: sample $(\boldsymbol{i}^*, \boldsymbol{j}^*)$ at random, then update

$$\mathsf{d}(\boldsymbol{i}^*, \boldsymbol{j}^*) \longleftarrow 1$$

Perturbed Path Metric

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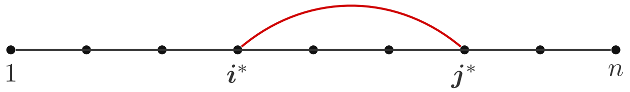
$\exists$ 1-navigable graph of degree 3:



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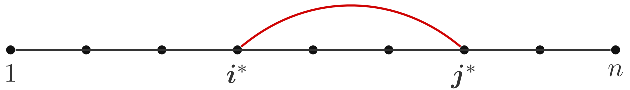

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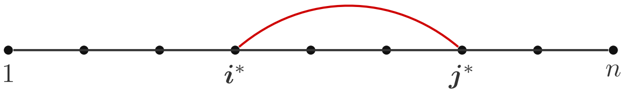

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[Conway et al. 2025] $\tilde{\Omega}(n^2)$ lower bound in Euclidean via Closest Pair

Navigability Landscape

Results 3, 4, 5

- $\tilde{O}(n \cdot \mathrm{OPT}_{\text{size}})$-time for $O(\ln n)$-approximation to SNG
- $\tilde{O}(n^{\omega})$-time for $O(\ln n)$-approximation to $(\alpha, 2\alpha)$-SNG
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Theorem [Conway et al. 2025]

- $\tilde{O}(n^2)$-time for $O(\ln n)$-approximation for $\alpha = 1$
- $\tilde{O}(n^{2.5})$-time for $O(\ln n)$-approximation for $\alpha > 1$

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Q. Is $\tilde{O}(n^2)$ time possible for $O(\ln n)$-approximation when $\alpha > 1$?

Thanks for Listening!